Destruction of stable spiral waves in oscillatory media

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We studied spiral wave dynamics in an oscillatory reaction diffusion system. We find a new phenomenon: without the appearance of any global modulation mode, stable spiral waves break up directly. By investigating the one-dimensional version of the system and the isolated local dynamics, we find that the unstable focus in the local dynamics plays an important role. For different boundary conditions (BCs), we find a transition between spiral waves and traveling waves for periodic BCs and drifting spiral wave for no-flux BC.

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Spiral waves are probably the most frequently encountered structures in nonequilibrium systems; their existence can be analytically proved even in the simplest λ-ω reaction diffusion systems [1,2]. Practically, these waves are observed in a wide range of systems, from chemical reactions and physiological media to slime mold aggregates and hydrodynamics. In mathematical language, spiral waves can be observed in excitable media, oscillatory media, and chaotic and stochastic media. Though a general theoretical understanding of these spiral waves has been emerging through decades of efforts, there are still significant gaps in our understanding of spiral dynamics.

The transition from a regular pattern to spatiotemporal chaos in extended systems remains a challenge in nonlinear dynamics [3]. Among these phenomena, the instability of spiral waves in reaction diffusion systems is one of the most robust scenarios observed in experiments and in numerical simulations [4–9]. The stability of stable spiral waves and the breakup of spiral waves have been widely studied in the literatures. Generally, stable spiral waves lose their stability by via a Hopf bifurcation that leads to meandering spiral waves [4]. There are two ways by which meandering spiral waves can go to break up. The first one is a longitudinal Eckhaus instability [5–8] that modulates the spiral waves, so that the minimum of the local period in the system violates the dispersion relation and results in breakup. Depending on the behavior of the Eckhaus mode, the spiral waves may break up far away from or near to the core [8]. The second breakup is caused by a transverse instability [9]. Both the breakup mechanism and the Hopf instability of the stable spiral wave require the appearance of a global mode. However, in some systems, we can find another mechanism where stable spiral waves break up directly without involving any global mode. In this paper, we study this case.

The model we used in this paper is a FHN-type reaction-diffusion system

$$\frac{\partial u}{\partial t} = -\frac{1}{\varepsilon}u(u-1)\left(u-\frac{b+v}{a}\right)+\nabla^2 u,$$
$$\frac{\partial v}{\partial t} = u-v, \tag{1}$$

where no-flux boundary conditions are used except where stated otherwise. In numerical simulations, discretizations with $dx=dy=0.166$ and $dt=0.00556$ have been used in an Euler scheme. The size of the system is 300×300. In the local dynamics, there are three fixed points: (*0*, *0*), (*1*, *1*) and $(b/(a-1),\ b/(a-1))$. When $b>0$, only (*0*, *0*) is stable, and the local dynamics is excitable. In this condition, the model has been studied as a candidate for the transition between stable spiral waves and meandering spiral waves [4], and in this case no spiral wave breakup is observed. However, when we change the parameter b to be negative, there is no stable fixed point. The saddle point $(b/(a-1),b/(a-1))$ becomes an unstable focus, and the local dynamics transitions to be oscillatory. The limit cycle that encompasses the unstable focus is confined in the area $(0,1)\times(0,1)$. Throughout this paper, we set $b=-0.02$ and $\varepsilon=0.02$. The spiral wave dynamics is shown in Fig. 1. Figure 1(a) shows a typical tip trajectory at $a=0.28$ in the meandering regime. The definitions of R_1 and R_2 are shown also in this figure. When $R_2=R_1$, the system has a stable spiral wave solution; otherwise it is in the meandering regime. In terms of R_1 and R_2, the amplitude of the Hopf mode could be obtained by $R=(R_1+R_2)/2$ when it is larger than the radius of the primary mode, otherwise it should be expressed as $R=(R_2-R_1)/2$. From Fig. 1(b), we know that a meandering spiral wave exists in the midrange of a investigated in this paper. There are two critical values of a, outside the interval between the two critical values:

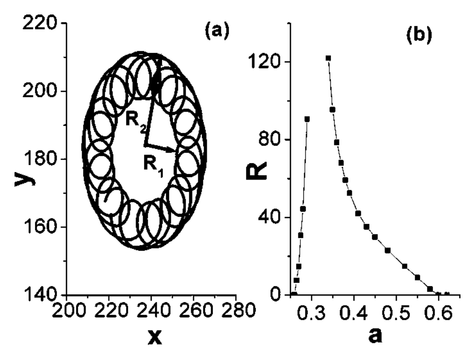

FIG. 1. (a) The tip trajectory at $a=0.28$, $b=-0.02$, and $\varepsilon=0.02$. The definition of R_2 and R_1 are shown. (b) The amplitude of the Hopf mode vs a with the same b and ε.

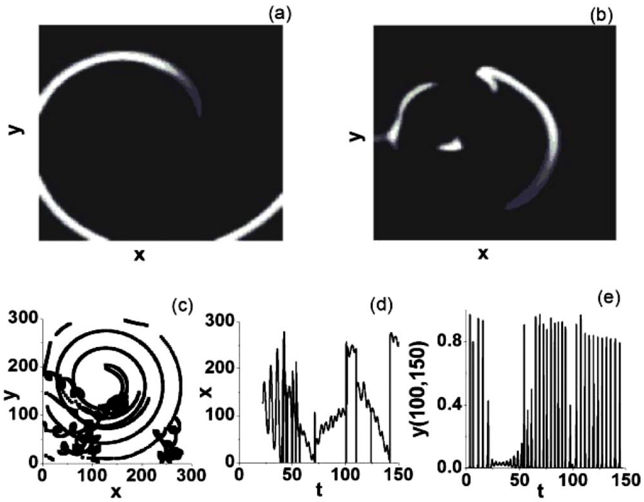

FIG. 2. The stable spiral wave directly breaks up at $a=0.252$. (a) The spatial plot of the spiral wave shortly after the spiral wave is initialized. The bright region indicates the high value of u. (b) The spatial plot where the spiral wave breaks up. (c) The tip trajectory. The tip spirals out along the previous spiral arm. (d) The evolution of the variable x at the tip. (e) Time sequence for site (100,150) located in the core region of initial spiral wave.

meandering spiral waves transition to stable spiral waves. We do not show the results in the middle part of the curve due to the fact that the size of the core of the spiral wave is comparable to or larger than the size of the system. The results are similar to Fig. 1 in Ref. [4] where $b>0$ is studied; for example, a meandering spiral wave with inward (outward) petal appears in the left (right) part of the meandering regime, and R grows exponentially toward the resonant point.

If we decrease a into the stable spiral wave regime, we find an interesting phenomenon different from what is seen in excitable media: the stable spiral wave cannot be sustained and it directly breaks up no matter how the initial spiral wave is generated. We show a case in Fig. 2. Figure 2(a) shows the spatial pattern soon after a spiral wave is generated. In this plot, the wave arm (bright area) decays gradually into the wave valley (dark area) not only in the direction of propagation (perpendicular to the wave front) but also in the transverse direction (tangential to the wave front) when the wave arm is near the tip of the spiral wave. As time goes on, the core of the spiral wave expands. Then after a while, the spiral wave is replaced by spatial-temporal chaos, which occurs first in the core region of the initial spiral wave [in Fig. 2(b)]. The power spectrum (not shown here) at any location is of broadband type.

To further describe the core expansion, we plot the tip trajectory in Fig. 1(c) and the time evolution of the x variable of the tip in Fig. 2(d) where the tip is defined by the isochrone $u=0.3$ (we only record one tip at any time even for the spatial-temporal chaos). The tip does not form a closed orbit and moves out along spiraling trajectories. The time sequence of a chosen site located in the core of the initial spiral wave is shown in Fig. 2(e). In Fig. 2(e) there are two kinds of oscillations observed: small amplitude oscillations and large amplitude oscillations. The small amplitude oscillation is faster than the large amplitude one. When the core of the initial spiral wave expands to the chosen site, the large amplitude oscillation at that site is replaced by the small amplitude one. Then the site stays at the small amplitude oscillation until its amplitude grows sufficiently large. Actually, the dark area left behind the tip [Fig. 2(a)] is full of such a small amplitude oscillation. New wave fronts are observed only when the amplitude of the oscillation is restored to the normal value. The sites in this dark area do not oscillate coherently; as a result spatial-temporal chaos will appear in this region. The outward movement of the tip of the initial spiral wave is caused by the collapse of the large oscillation to the small one that cannot send the system back to its normal value immediately.

The similar core expansion was investigated by Meron [10] and Sabbagh [11] in excitable media. The theory [10] presented by Meron attributes the core expansion to the wave front interactions in an oscillatory recovering medium. Sabbagh [11] found that the core expansion in a modified version of Barkley's standard model. However, Sabbagh also pointed out that the oscillatory or damped oscillatory character in a dispersion curve would not necessarily lead to core expansion. That is, the mechanism of the core expansion is still unknown. Due to the nature of the oscillatory medium, the core expansion observed in this paper is in the oscillatory recovery, nut not in the damped recovery as excitable medium does. There is another important difference to be addressed. In the Sabbagh's case, with the change of the controlling parameter, stable spiral waves first transition to meandering spiral waves, then to the core expansion. However, in our paper the core expansion occurs after meandering spiral waves transition to stable spiral waves. Therefore, the investigation of the mechanism of the core expansion is not trivial.

The spatial-temporal chaotic motion shown in Fig. 2(b) is only a transient process. Depending on the boundary conditions, the system has different fates. First, under a no-flux boundary condition, we use the S1-S2 method to initiate a spiral wave [Fig. 3(a)]. After a long run of transient spatiotemporal chaos, a spiral wave with its core near the boundary is observed [Fig. 3(c)]. The tip trajectory in Fig. 3(d) shows that the spiral wave is drifting along the boundary. The radius of the tip of the drifting spiral wave is around 20, which is nearly the same as that for $a=0.254$, but the frequency of the spiral wave is higher than that for $a=0.254$ because of its drifting. Usually the drifting of a spiral wave is caused by an external field [12] or by the interaction between spiral waves. Here the drifting of single spiral wave is induced by boundary [13,14]. However, when we change the boundary condition to be periodic, a transition from a spiral wave to a traveling wave is observed [Fig. 3(b)]. It is necessary to mention that the total topological charge of the spiral waves under periodic boundary conditions is zero. This conclusion can be confirmed by the fact that the spiral wave can only be generated or annihilated in pairs with opposite charge (positive or negative). The orientation of the traveling wave depends on the initial condition.

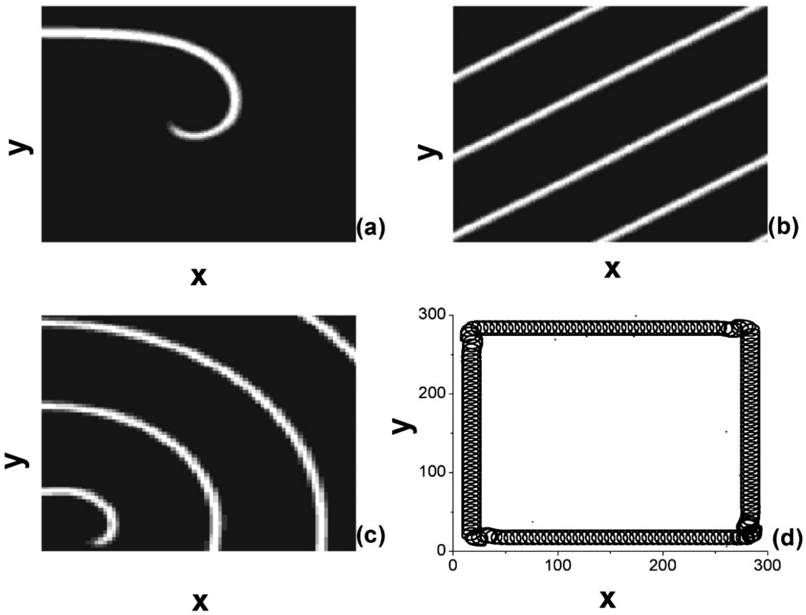

FIG. 3. Different fates of the spiral wave when the boundary condition changes at $a=0.252$. (a) The initial spiral wave. (b) A traveling wave is realized under periodic boundary condition. (c) Drifting spiral waves anchoring the boundary under no-flux boundary condition. (d) The tip trajectory under the no-flux boundary condition.

Actually, the small amplitude oscillation that results in the destruction of the stable spiral wave has its root in the isolated local dynamics. We already mentioned that there exists an unstable focus in the local dynamics. Its eigenvalues can be expressed as

$$\lambda=[x-1\pm\sqrt{(x-1)^2+4(x+y)}]/2,$$

$$x=-\frac{b(b-a+1)}{\varepsilon(a-1)^2},\quad y=-x/a.$$

We plot the real and imaginary parts of the eigenvalue versus a in Fig. 4. The real part of the eigenvalue describes the rate of the deviation away from the unstable focus. It increases with the increase of a. The imaginary part describes the angular velocity of the deviation around the unstable focus. This velocity is much faster than the asymptotic limit cycle shown in Fig. 4(c). With an initial condition close to the unstable focus, the isolated site will rotate several times around the unstable focus before it reaches the asymptotic limit cycle. The number of rotations before reaching the limit cycle increases as the initial deviation decreases, and the time spent on one rotation increases as the deviation increases. Figure 4(d) shows one example.

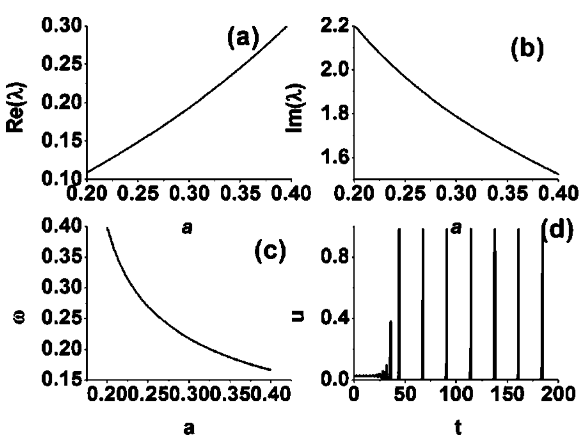

FIG. 4. (a) The real part of the eigenvalue vs a. (b) The imaginary part of the eigenvalue vs a. (c) The angular velocity of the isolated local dynamics. (d) The time evolution with a small initial deviation from the unstable focus.

To further investigate the mechanism that leads stable spiral waves to be destroyed, we study the one-dimensional (1D) version of Eq. (1). The rightmost end obeys a no-flux boundary condition while the leftmost end is fixed at the unstable focus $[b/(a-1),\ b/(a-1)]$. The discretizaion is the same as that in the 2D simulation, and the size of the system is 200. Actually for the stable spiral wave, its rotation center is time dependent. Within the parameter range we studied, the rotation center falls on the unstable focus in this system. So the 1D system here describes the radial dynamics of stable spiral waves that emit waves from the rotation center. For the sake of simplicity, we do not consider curvature effects in this paper. Due to continuity, the fixed value driving force has to drag nearby sites along with it. Once the state of the neighbor site is close enough to its unstable focus, it will take some time to spiral out of it. Such an effect becomes stronger when a decreases. In Fig. 5, we show the transition from regular dynamics to irregular one caused by this effect. When a is large [$a=0.337$ in Fig. 5(a)], the wave emitted by the fixed value boundary is propagating downstream with a constant period; the frequency is around 0.16. However, when we decrease a beyond a critical value around 0.337, the steady traveling wave cannot be sustained at a constant period and the irregularity first appears near the source. The time evolution for site No. 10 is shown in Figs. 5(c) and 5(d) for $a=0.337$ and 0.336, respectively. In Fig. 5(c), after a short transient, the site reaches its asymptotic periodic oscillation. In Fig. 5(d), the site first oscillates at small amplitude for a while, then jumps to large amplitude oscillations. After that, the large amplitude oscillation and the small amplitude oscillation appear alternately at random. The life span of the small amplitude oscillation decreases

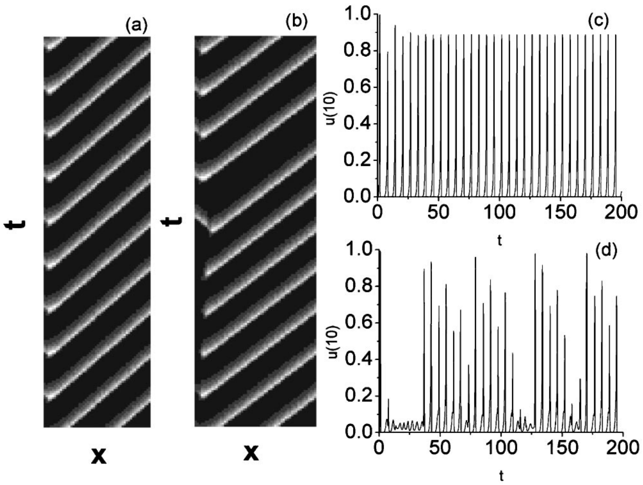

FIG. 5. The dynamics of the 1D system. (a) The spatial-temporal plot for $a=0.337$. No irregularity sets in. (b) The spatial-temporal plot for $a=0.336$. The irregularity is observed. (c) The time evolution of site #10, $a=0.337$. (d) The time evolution of the same site, $a=0.336$.

with the increase of the distance away from the driving source.

From the spatial-temporal dynamics in the 1D case, we know that the rotation center of the stable spiral wave cannot maintain a periodic wave any longer when a is small enough. The rotation center has to induce the small oscillation first in the core region, and finally leads to the destruction of the stable spiral wave.

In the 2D medium, the meandering spiral wave is stable for $0.6>a>0.255$. In this regime, the description of the 1D medium with a fixed value boundary condition is not valid. Actually, to describe the meandering spiral wave, the fixed value boundary condition has to be replaced by a quasiperiodic driving force at the leftmost end. The simulations of a 1D system with the quasiperiodic driving force do not show any irregularity in the corresponding range of a using the frequencies obtained from the 2D simulations. Therefore, though a traveling wave with a constant period cannot be sustained below $a=0.336$ in the 1D dynamics, there is no irregularity appearing in the 2D system yet. When the system transitions from a meandering spiral wave to a stable spiral wave around $a=0.255$, the stable spiral wave breaks up immediately. Similar to a meandering spiral wave, a drifting spiral wave, evolved from a stable spiral wave under a no-flux boundary condition, cannot be described by the 1D system with fixed value driving force either. Instead, its 1D version is that with a periodic driving force.

The direct breakup of the stable spiral wave, according to the mechanism discussed above, only occurs when the rotation center of the spiral wave is located at the unstable focus. However, it is not limited to oscillatory media. In fact, even in excitable systems, the phenomenon we observed in this paper could be found if the isolated dynamics has a unstable focus.

In summary, we studied the spiral dynamics in an oscillatory reaction diffusion system. We find a new phenomenon: without the appearance of any global modulation mode, the stable spiral wave can directly break up. By investigating the 1D version of the system and the isolated local dynamics, we found that the unstable focus in the local dynamics plays a very important role. It causes small amplitude oscillations that finally lead the initial stable spiral wave to break up. We also find a transition between a spiral wave and a traveling wave under periodic boundary conditions.

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